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Spectral Automorphisms in Quantum Logics

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Abstract In quantum mechanics, the Hilbert space formalism might be physically justified in terms of some axioms based on the orthomodular lattice (OML) mathematical structure (Piron in Foundations of Quantum Physics, Benjamin, Reading, 1976). We intend to investigate the extent to which some fundamental physical facts can be described in the more general framework of OMLs, without the support of Hilbert space-specific tools. We consider the study of lattice automorphisms properties as a "substitute" for Hilbert space techniques in investigating the spectral properties of observables. This is why we introduce the notion of spectral automorphism of an OML. Properties of spectral automorphisms and of their spectra are studied. We prove that the presence of nontrivial spectral automorphisms allow us to distinguish between classical and nonclassical theories. We also prove, for finite dimensional OMLs, that for every spectral automorphism there is a basis of invariant atoms. This is an analogue of the spectral theorem for unitary operators having purely point spectrum.

Keywords Orthomodular lattice · Quantum logic · Lattice automorphism · Nonclassical theories

1 Introduction

In quantum mechanics, the Hilbert space formalism might be physically justified in terms of some axioms based on the orthomodular lattice (OML) mathematical structure [6]. Since the framework of orthomodular lattices/quantum logics is, in a sense, more general than

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that of Hilbert space, which is supported by specific mathematical structures, we intend to investigate what amount of "quantum physics" is already contained in the mathematical structure of orthomodular lattices. In other words, the problem is to have some look on the dependence of several fundamental physical facts on Hilbert space specific tools.

Towards this end, we introduced the notion of spectral automorphism and studied their fundamental mathematical properties and connections with some quantum problems.We have also used spectral automorphisms in our attempt to clarify the physical meaning of some currently used Hilbert-space mathematical objects.

A very well known result in the theory of Hilbert space [1, 8] states that there exists a one to one correspondence between three sets:

- the set of selfadjoint operators
- the set of spectral measures
- the set of all one-parameter strongly continuous groups of unitary operators.

This observation is consistently used in the following considerations.

Before the definition enounced, let us explain the motivation and the origin of the notion of spectral automorphism. One of the most fundamental objects in the Hilbert-space quantum theory is that of the spectrum of an observable/selfadjoint operator. Its definition depends strictly on the fact that the observable in the Hilbert-space framework is a linear operator in a vector space endowed with a topology generated by a scalar product. On the other hand—and this is crucial for our approach—observables in orthomodular lattices are quite similar as mathematical objects to spectral measures, which define the Hilbert-space observables/selfadjoint operators. However, a notable difference consists in the fact that the spectral measures have as values orthogonal projectors in a Hilbert space but any observable of an orthomodular lattice has as values elements of that orthomodular lattice. This observation leads to the idea that considering the properties of lattice—automorphisms might be a "substitute" of Hilbert-space technique in studying the spectral properties of observables in the more general framework of orthomodular lattices. In the next section, we shall see more precisely where this idea comes from.

Before concluding this section, let us summarize some of the definitions and notations that will be used in our exposition. For an in-depth discussion of these notions and of their mutual relations, we refer the reader to e.g. [3] or [4].

In what follows, $(L, \leq, \perp)$ will be an orthomodular lattice. We shall also assume L to be *complete*, as a lattice. For simplicity, we will sometimes write L instead of $(L, \leq, \perp)$. Two elements $a, b \in L$ *commute* (or are *compatible*) if $a = (a \wedge b) \vee (a \wedge b^{\perp})$. In an OML, commutativity is symmetric. It is also noteworthy that the necessary and sufficient condition for an OML to be a Boolean algebra is that every pair of its elements commutes. The *center* of an OML is the set of its elements that commute with all the other elements of that OML. If M is a subset of L, the *commutant* of M (in L) is the set of elements of L that commute with all the elements of M. A maximal subalgebra of mutually commuting elements of L is called a *block*. Therefore, considering the previous remarks, a block is a maximal Boolean subalgebra of L. An element $\alpha \in L$ is called an *atom* of L if it is a minimal nonzero element. We call L *atomic* if any nonzero element of L is greater than (or equal to) an atom of L. For $a, b \in L$, we shall say that b *covers* a and write $a \lessdot b$ if $a < b$ and $a < c < b$ is not satisfied by any c. L is said to have the *covering property* if for any $a \in L$ and p atom of L such that $a \wedge p = 0$, we have $a \lessdot a \vee p$. A *base* of an element $a \in L$ is a maximal set of mutually orthogonal atoms smaller than a. The following notations will be used:

1. $a \perp b$ if a and b are orthogonal (i.e. $a \leq b^{\perp}$);
2. $(a, b)K$ if a, b commute/are compatible;

3. $(a, M)K$ iff $(a, b)K$, $(\forall)b \in M \subseteq L$; similarly $(M, N)K$ for $M, N \subseteq L$;
4. $K(M)$ for the commutant of $M \subseteq L$;
5. $C(F)$ for the center of the orthosublattice/subalgebra $F \subseteq L$;
6. $\Omega(F)$ for the set of all atoms of the subalgebra $F \subseteq L$;
7. $[M]$ for the subalgebra generated by $M \subseteq L$.

2 Spectral Automorphisms. General Properties

Let us continue now the reasoning we started in the previous section, in order to motivate our definition of a spectral automorphism. Suppose we have a Hilbert space H and a selfadjoint operator A on H. We can then construct an operator, denoted by $U_A = e^{iA}$, which, according to the rules of functional calculus, is unitary. Moreover, it defines an automorphism on the orthomodular lattice $\Pi(H)$ of all orthogonal projectors on the space H. To be more precise U_A defines an application $E \longmapsto U_A E U_A^{-1}$ for all $E \in \Pi(H)$, which is obviously an automorphism of $\Pi(H)$. Therefore the following idea appears natural: if it is desired to obtain something "similar" to the spectral theory of selfadjoint operators in orthomodular lattices, then a way to do this might be related to the lattice automorphisms.

Let V be the automorphism on $\Pi(H)$ defined by $E \longmapsto U_A E U_A^{-1}$ for all $E \in \Pi(H)$ and let B_U denote the Boolean algebra of projectors that is the image of the spectral measure associated to U_A. A projector $E \in \Pi(H)$ is invariant under V if and only if $EU_A = U_A E$, i.e. the two operators commute. According to a well known result of the spectral theory in Hilbert space [2], this happens if and only if E commutes with B_U (i.e. commutes with every projector in B_U). So finally $E \in \Pi(H)$ is V-invariant if and only if E commutes with a Boolean subalgebra of $\Pi(H)$. Abstracting this to the more general framework of orthomodular lattices suggests the following:

Definition 2.1 An automorphism $V : L \longrightarrow L$ is said to be *spectral* if there exists B a Boolean subalgebra of L with the property

$$V(a) = a \quad \Longleftrightarrow \quad (a, B)K \tag{P1}$$

Let L_V denote the set of all V-invariant elements of L .

Proposition 2.1 *L_V is a subalgebra of L.*

Proof The statement is a straightforward consequence of the properties of V. □

Before we can go further to define the spectrum of a spectral automorphism and study its properties, we need to recall some of the well established facts about commutants and Boolean subalgebras in OMLs. We shall omit the proofs of these results, which can be found in classical monographs like [3] or [9].

Let M, N be subsets of L. It is then easy to see that $M \subseteq K(K(M))$ and $M \subseteq N \Longrightarrow K(N) \subseteq K(M)$.

Proposition 2.2 *Let M be a subset of OML L. Then $K(M)$ is a subalgebra of L.*

Proposition 2.3 *Let M be a subset of L. Then $M \subseteq K(M) \Longleftrightarrow M \subseteq B$ for some block B of L.*

It follows that $B = K(B)$ for any block B of L. Also, L is the set-theoretical union of its blocks and $C(L)$, the lattice center, is their intersection.

In any Boolean algebra, any two distinct atoms are orthogonal. This is a straightforward consequence of the fact that they commute. From this, the following assertion can be deduced easily:

Proposition 2.4 *If B is a Boolean subalgebra of L, $\omega \in L$ and $\omega \leq a \in \Omega(B)$, then $(\omega, B)K$*

Proposition 2.5 *Let B be a block of L. Then the atoms of B are also atoms of L.*

The following result is from [9]:

Proposition 2.6 *Let $\{B_i\}_{i\in I}$ an arbitrary family of Boolean subalgebras of L. Then $[\bigcup_{i\in I} B_i]$ is a Boolean subalgebra of L iff $(B_i, B_j)K$ for any $i, j \in I$.*

We shall now return to our investigation of spectral automorphisms. The next proposition is very important, since it leads to the definition of the spectrum of a spectral automorphism.

Proposition 2.7 *For any spectral automorphism, there exists a greatest Boolean subalgebra having the property* (P1).

Proof Let V be a spectral automorphism and $\{B_i; i \in I\}$ the set of all Boolean algebras having the property (P1) with respect to V. Then for any $i, j \in I$, $i \neq j$ we have $(B_i, B_j)K$. Indeed, $a \in B_i \Longrightarrow (a, B_i)K \Longrightarrow V(a) = a \Longrightarrow (a, B_j)K$. According to Proposition 2.6, $[\bigcup_{i\in I} B_i]$ is a Boolean algebra and it satisfies (P1), as one can easily check, by applying the general properties of commutativity (see, e.g. [4]). □

Definition 2.2 If $V : L \longrightarrow L$ is a spectral automorphism, the greatest Boolean subalgebra having the property (P1) is called the *spectrum* of V and will be denoted by σ_V.

A trivial example shows that in general there are several Boolean subalgebras having the property (P1) with respect to a given automorphism. Indeed, let L be a Boolean algebra and $\mathbf{1}_L : L \longrightarrow L$ its identity. Obviously, $\mathbf{1}_L$ is an automorphism, and any Boolean subalgebra of L has the property (P1). It follows that $\mathbf{1}_L$ is spectral and its spectrum is L. We can prove even more:

Proposition 2.8 *If L is a Boolean algebra, then $\mathbf{1}_L$ is the only spectral automorphism of L.*

Proof Let V be a spectral automorphism of L and σ_V its spectrum. Since L is a Boolean algebra, we obviously have $K(\sigma_V) = L = L_V$. Therefore, all elements of L are invariant under V, so that $V = \mathbf{1}_L$. □

A simple, but very important consequence of this fact is:

Corollary 2.1 *If an OML has a nontrivial spectral automorphism, then it cannot be Boolean.*

The following proposition is a characterization of spectral automorphisms and their spectra.

Proposition 2.9 *The automorphism $V : L \longrightarrow L$ is spectral if and only if there is a Boolean subalgebra B of L such that $L_V = K(B)$. In this case, $\sigma_V = C(L_V)$.*

Proof The first statement is an obvious reformulation of the property (P1). If V is spectral, we have $L_V = K(\sigma_V)$. Therefore, any element of σ_V commutes with all elements of L_V, which means that $\sigma_V \subseteq C(L_V)$. On the other hand we may easily verify that $C(L_V)$ has the property (P1). Since the spectrum of V is the greatest Boolean algebra with this property, the second assertion is proved. □

Let us consider an automorphism V and denote, for shortness $C(L_V) = C_{V.}$ Since it's obvious that $L_V \subseteq K(C_V)$, the following corollary results:

Corollary 2.2 *The automorphism $V : L \longrightarrow L$ is spectral if and only if $K(C_V) \subseteq L_V$.*

This means that L_V must be "sufficiently large" for including the commutant of it's center. Similarly, we might say that the center C_V must have "sufficiently many" elements for its commutant to be included in the lattice of invariant under V elements.

3 C-maximal Boolean Subalgebras of an OML

We intend to look into the following matter: what could be the condition for a Boolean subalgebra of an OML L to be the spectrum of an automorphism of L? It is clear, considering the previously stated results, that for this to happen, the Boolean algebra B must satisfy $B = C(K(B))$. Since we always have $B \subseteq C(K(B))$, the condition reduces to the inverse inclusion. Therefore, we give the following:

Definition 3.1 A Boolean subalgebra $B \subseteq L$ satisfying $C(K(B)) \subseteq B$ is said to be *C-maximal* (i.e. maximal with respect to its commutant).

As we already mentioned, C-maximality is a necessary condition for a Boolean subalgebra of an OML to be the spectrum of an automorphism of that OML.

Example 3.1 Any maximal Boolean subalgebra (block) of an OML is C-maximal, since it coincides with its commutant. On the other hand it is simple to find Boolean subalgebras which are not maximal. Indeed, let us consider L an OML having only two blocks, B_1, B_2 and having also the property $B_1 \cap B_2 \neq \{0, 1\}$. Obviously, $C(L) = B_1 \cap B_2$ and $K(\{0, 1\}) = L$. Therefore, $\{0, 1\}$ is not C-maximal.

Theorem 3.1 *A Boolean subalgebra is C-maximal if and only if it coincides with its bicommutant.*

Proof Suppose $B = K(K(B))$. This means that B is the set of *all* elements commuting with $K(B)$. Since $B \subseteq K(B)$ it is the center of $K(B)$. Therefore, $B = C(K(B))$ which means it's C-maximal.

Conversely, if B is C-maximal, we need to prove $K(K(B)) \subseteq B$, since the inverse inclusion is trivial. Let $b \in K(K(B))$. Then we can write:

$$(b, K(B))K \tag{1}$$

Since $B \subseteq K(B) \Longrightarrow$ (according to (1)) $(b, B)K$, and so

$$b \in K(B) \tag{2}$$

Now, if we put together (1) and (2), we deduce that $b \in C(K(B)) = B$, which concludes the proof. □

Theorem 3.2 *If the automorphism V is spectral, then the following assertions are true*:

(i) $C_V = K(L_V)$
(ii) C_V *is C-maximal.*

Proof Considering that $C_V = C(L_V)$, it's obvious that $C_V \subseteq K(L_V)$. We shall prove the inverse inclusion. Let $a \in K(L_V)$. Then $(a, L_V)K \Longrightarrow (a, \sigma_V)K$ (since $\sigma_V \subseteq L_V$). But according to the definition property of σ_V, this means $a \in L_V$, hence $a \in C(L_V) = C_V$. This concludes the first part of the proof.

The second assertion results easily because of Corollary 2.2. We have $L_V = K(C_V)$ and hence $C_V = C(L_V) = C(K(C_V))$. Therefore, C_V is C-maximal. □

4 Spectral Automorphisms and Physical Theories

Since orthomodular lattices have been considered in this work—and not only—the roots of any physical theory, let us examine the importance of spectral automorphisms for some fundamental problems concerning their classification and also their structure.

Let us begin by reminding that in our discourse a physical theory is an orthomodular lattice, which is occasionally assumed possessing some supplementary properties, like having atoms, satisfying the covering law or being complete.

Let us see now what is the essential difference between classical and non-classical physical theories from the point of view of spectral automorphisms. In Proposition 2.8, it has been shown that a classical theory/Boolean algebra has no nontrivial spectral automorphisms, that is, the only spectral automorphism is the identity of the theory in question. So if a theory has at least one nontrivial spectral automorphism, it has to be non-classic.

We do not intend to prove here very general results concerning non-classical theories, but we will show that in such theories there are states, which are invariant under spectral automorphisms. For doing this there will be considered some particular cases of spectral automorphisms and non-classical theories.

Let us consider a theory L, which is atomic, complete and has the covering property. Assume also that this theory has a spectral automorphism V whose spectrum is atomic. Below we will show that examples of such theories exist, such as a finite dimensional quantum logic. Then, given $a \in \Omega(\sigma_V)$ and $\omega \in \Omega(L)$, $\omega \leq a$, by Proposition 2.4, $\omega \in K(\sigma_V)$. Moreover, V being spectral, it follows that ω is invariant under V. Since L is atomic, complete and has the covering property, it follows that each $a \in \Omega(\sigma_V)$ has a basis of atoms of L, let us denote it by B_a [4]. The conclusion is that $\bigcup_{a\in\Omega(\sigma_V)} B_a$ is a basis of L, whose elements are invariant under V. It is almost obvious that this result is an analogue of the spectral theorem for unitary operators having purely point spectrum and also for the corresponding selfadjoint operator. If L has a finite dimension, it is easy to probe that all spectral automorpisms have atomic spectra, which means that the just proved result may be applied with no modifications.

5 Spectral Automorphisms and Piron's Theorem

Piron's representation theorem [5] is still the only fundamental result allowing us to consider that non-classical theories are based on the Hilbert space formalism. However, a very delicate problem remains to discuss: is the Hilbert space real, complex or quaternionic? This problem is, in a sense, still open. Our intention is to prove that choosing one of the first two possibilities depends strongly on a property of the symmetries of the theory in question.

Let us remind a very important result concerning physical theories: any symmetry of a physical theory is represented by an automorphism of that theory [6]. Now if we apply the last result of the previous section to spectral symmetries (i.e. symmetries that are represented by automorphisms that are spectral) we reach the following interesting conclusion: if admitted that a finite dimensional physical theory must have spectral symmetries, its representation by the lattice of projectors of a finite dimensional *real* Hilbert space is impossible. Indeed, a well-known theorem affirms that no symmetry in such a space has a complete system of invariant one-dimensional subspaces (see, for instance, [7]). Therefore, if there are physical motivations for admitting that spectral symmetries must exist in a theory, then the real Hilbert spaces have to be excluded from those able to support quantum theories.

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